Chaos Control and Anti-control in Fractional Order Rössler System by Parameter Switching Method

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Resumen

Este artículo describe la metodología para llevar a cabo la supresión y el control de caos en un sistema de orden fraccional. El caso de estudio es el sistema de Rössler de orden fraccional. La dinámica del sistema se examina primero con el diagrama de bifurcación y el espectro de Lyapunov, lo más importante para identificar las ventanas estables y las regiones caóticas. El control y anti-control del caos en el orden fraccional del sistema Rössler se logra aplicando el esquema de cambio de parámetros utilizando el parámetro *c* del sistema como parámetros de cambio p. En el control del caos, la técnica de cambio de parámetros utiliza atractores caóticos para sintetizar un ciclo estable, mientras que los ciclos estables se utilizan para sintetizar un atractor inestable en el caso de control. En ambos casos, la solución "conmutada" del esquema de conmutación de parámetros se compara con el atractor subyacente de la solución "promediada" obtenida por el p^* promedio de los valores conmutados. Los resultados muestran que la solución "conmutada" representa la aproximación de la solución "promediada". Los resultados se verifican usando retratos de fase, exponentes de Lyapunov y respuestas en el dominio del tiempo. Todas las simulaciones en este trabajo se llevaron a cabo en MATLAB.

Palabras clave — Anti-control del caos, Control del caos, Sistema Rössler de orden fraccional, Exponente de Lyapunov, Conmutación de parámetros.

Abstract

*This paper describes the methodology to carry out the suppression and anti-control of chaos in a fractional order system. The case study is fractional order Rössler system. The dynamics of the system are first examined with bifurcation diagram and Lyapunov spectrum, most importantly to identify the stable windows and chaotic regions. Control and anti-control of chaos in the fractional order Rössler system is achieved by applying parameter switching scheme using the system parameter c as the switching parameters **p**. In chaos control, the parameter switching technique uses chaotic attractors to synthesize a stable cycle, while stable cycles are used to synthesize an unstable attractor in the anti-control case. In both cases, the "switched" solution of the parameter switching scheme are compared with the underlying attractor of the "averaged" solution obtained by the average* p^* *of the switched values. The results show that the "switched" solution represents the approximation of the "averaged" solution. The results are verified using phase portraits, Lyapunov exponents, and time domain responses. All simulations in this work were carried out in MATLAB.*

Keywords *— Anti-control of chaos, Chaos control, Fractional order Rössler system, Lyapunov exponent, Parameter switching.*

1. INTRODUCTION

In the last few decades, the study of non-linear dynamics has given rise to a whole new perspective known as chaos. Chaos theory, also known as the "Butterfly Effect," has applications in several sectors such as telecommunication [1, 2], engineering [3, 4], and many more. Several studies have been conducted to examine the dynamic behavior in chaotic oscillators, including chaos control in the systems.

Chaos control is a subject of interest in chaos theory. It means suppressing chaos by stabilizing chaotic system responses, or transition between chaos and order. The concept is applicable in designing transmission systems whereby transmitted information, e.g. RGB image, can be recovered by stabilizing the chaotic system. On the other hand, anti-control of chaos makes a non-chaotic dynamical system chaotic or retains existing chaos in chaotic systems. It has a useful potential in system control theory.

The authors in [5] studied the stabilization of a chaotic fractional order generalized Lokta-Volterra (GLV) model. The article in [6] presented the work on chaos control of Burke-Shaw system using time delayed feedback control. In [7] parameter switching technique was used to control chaos in non-commensurate fractional order Chen oscillator. Chaos in a fractional order system with coexisting attractors was studied in [8] and controlled using a single state variable linear controller. Fractional order financial system was examined in [9] for chaos control and anti-control by parameter switching.

In this work, the objective is the control and anti-control of chaos in commensurate fractional order Rössler system. The main activity of this paper involves the application of parameter switching methodology to achieve the stated objective. Unlike other stabilization methods such as OGY, which involves "forcing" unstable periodic orbits to become stable, parameter switching enables convenient generation of any desirable stable attractor.

The paper is organized in sections as follows. Section 2 contains the theoretical framework of the investigation. The results of the work are presented in Section 3, while the conclusion of the paper is in Section 4.

2. THEORETICAL FRAMEWORK

This section comprises the description of the chaotic system used as case study in this paper, namely, fractional order Rössler system. Also, the parameter switching technique applied for control and anti-control of chaos in the afore-mentioned fractional system is presented.

2.1 Fractional Order Rössler System

The Rössler system, described in [10], was created by Otto Rössler as a continuous-time dynamical system that exhibits a single lobe chaotic attractor. It comprises of the following system of three non-linear ordinary differential equations:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \qquad [1]$$

where *x, y,* and *z* are the system dependent variables and *a*, *b* and *c* are the system parameters. The system is chaotic when $a = 0.1$, $b = 0.1$, and $c = 14$.
The autonomous fractional order variant of the Rössler system above, modeled by the initial value problem (IVP) with Caputo's derivative, is of the form:

$$D_*^q x(t) = f\big(x(t)\big), \qquad x(0) = x_0, \qquad t \in [0, T] \qquad [2]$$

where D_*^q is the Caputo's deferential operator of order $0 < q \leq 1$ and $q=[q_1, q_2, q_3, \ldots, q_n]^T$, $f\colon \boldsymbol{R^n} \to \boldsymbol{R^n}$ is a Lipschitz continuous nonlinear function, $x_0 \in \boldsymbol{R^n}$ is the initial condition and $T > 0$. If $q_1 = q_2 = q_3 = \ldots = q_n$, the fractional system is called commensurate order, otherwise, non-commensurate order. D_*^q is defined by:

$$D_*^q x(t) = \frac{1}{\Gamma(m-q)} \int_0^t \frac{x^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, \quad m - 1 < q < m \qquad [3]$$

where *m* is the smallest integer larger than *q*, $x^{(m)}(t)$ is the *m*-order derivative in the usual sense and Γ is the Euler's Gamma function.
Therefore, the fractional order equivalent of the Rössler system stated above is given in the next equation:

$$\begin{aligned} D_*^q x &= -y - z \\ D_*^q y &= x + ay \\ D_*^q z &= b + z(x - c) \end{aligned} \qquad [4]$$

which is chaotic when $a = 10$, $b = 40$, $c = 2.5$, $h = 4$ and $k = 1$. When $q = 1$, the system becomes the original Rössler system.

2.2 Parameter Switching Technique

The parameter switching (PS) is an elegant method for approximating numerically any attractor of a dynamical system modeled by a general IVP of integer and fractional order. It involves selecting a finite set of parameter values in which the control parameter *(p)* within the chosen set is switched in some periodic (deterministic) manner for relatively short time subintervals, while the underlying IVP is numerically integrated [7]. At the end, the solution obtained from the "switched" system of the PS scheme will approximate the one obtained when the parameter is replaced with the average (p^*) of the switched values.
Given the following general IVP of fractional order,

$$D_*^q x(t) = f\big(x(t)\big) + pAx(t), \quad x(0) = x_0, \quad t \in [0, T] \qquad [5]$$

where D_*^q is the Caputo's deferential operator of order $0 < q \leq 1$, $f\colon \boldsymbol{R^n} \to \boldsymbol{R^n}$ is a Lipschitz continuous nonlinear function, $p \in R$ is the control parameter, $x_0 \in \boldsymbol{R^n}$ is the initial condition, $T > 0$ and $A \in L(R^n)$ is a constant matrix, the PS algorithm is expressed in the scheme that follows:

$$[m_1 p_1, m_2 p_2, \ldots, m_N p_N] \qquad [6]$$

where p_i are the control parameters, i.e. *p* in the general IVP above, m_i represent the weights associated with each p_i and $N > 1$. While the IVP is integrated, for m_1 integration steps $p = p_1$, for the next m_2 steps, $p = p_2$, and so on until the last m_N steps when $p = p_N$. Following the same procedure, the PS algorithm repeats for the next set of *N* values of *p* until the integration time interval is covered.
Applying the PS scheme, the obtained "switched" solution of the IVP will converge to the "averaged" solution obtained for $p = p^*$, which is calculated thus:

$$p^* = \frac{\sum_{i=1}^{N} m_i p_i}{\sum_{i=1}^{N} m_i} \qquad [7]$$

where p^* is the average parameter.

Let $\mathcal{A}$ denotes the set of attractors depending on *p*, $\mathcal{P}$ is the set of all possible *p* values, $\mathcal{P}_N = \{p_1, p_2, \ldots, p_N\} \subset \mathcal{P}$ is the set of chosen *p*, $\mathcal{A}_N = \{A_{p1}, A_{p2}, \ldots, A_{pN}\} \subset \mathcal{A}$ is the set of attractors corresponding to $\mathcal{P}_N$, A^S denotes the synthesized attractor obtained from the "switched" solution using PS scheme, and A^* is the attractor of the "averaged" solution when $p = p^*$.

3. RESULTS

First and foremost, the dynamics of the fractional order Rössler system are examined with bifurcation diagram and Lyapunov exponent (LE) spectrum against the varied value of the system parameter *c*. The bifurcation diagram and LE spectrum give insights into the qualitative behavior of the system as the parameter value is varied.
In the application of parameter switching technique to the general IVP of fractional order stated above for control and anti-control of chaos in the fractional order Rössler system,

the number of control parameter p in the IVP is N = 4. Therefore, the corresponding PS scheme is:

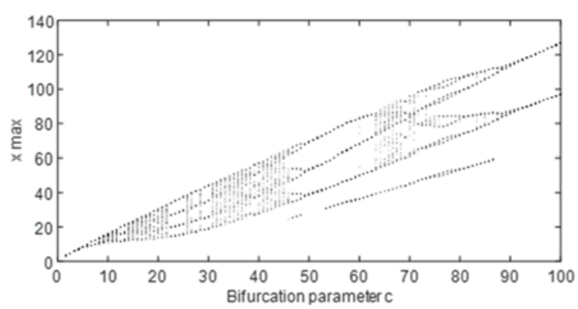
$$[m_1p_1, m_2p_2, m_3p_3, m_4p_4] \qquad [8]$$

while $\mathcal{P}_4 = \{p_1, p_2, p_3, p_4\}$ and $\mathcal{A}_4 = \{A_{p1}, A_{p2}, A_{p3}, A_{p4}\}$.

The underlying IVP is integrated with a time interval of T = [0,700] while the integration step size h = 0.001. The fractional order Rössler system in this paper is of commensurate order in which p_1=p_2=p_3=0.9999. The convergence of PS algorithm is verified for approximating the solution from the average parameter p^* by comparing the attractor A^S (plotted in red) with attractor A^* (plotted in blue). All simulations were carried out in MATLAB 2016b on the following computer configuration: Processor: Intel(R) Core(TM) i7-4790, 3.60GHz; RAM: 12 GB; Operating System: Windows 10. The bifurcation diagram of state x and LE spectrum against the varied value of system parameter c (i.e. control parameter p) while keeping parameters a and b constant are shown in Figure 1.

Fig. 1. Dynamics of Fractional order Rössler system (a) Bifurcation diagram of state x (b) Lyapunov exponents spectrum

(a)



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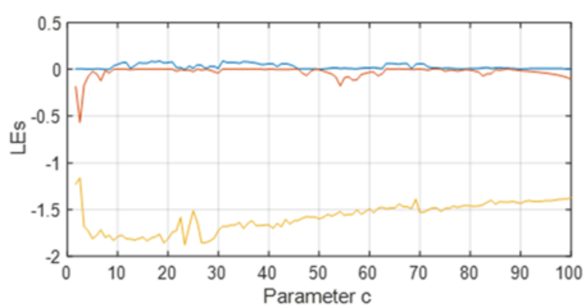


3.1 Case 1: Chaos Control

In the bifurcation diagram presented in Figure 1, the focus is on the stable window where control parameter p = 23.33. To obtain the stable periodic cycle based on PS scheme $[m_1p_1, m_2p_2, m_3p_3, m_4p_4]$, let p_1 = 14, p_2 = 18.33, p_3 = 30.83, p_4 = 32.66, m_1 = 1, m_2 = 3, m_3 = 2, and m_4 = 1. The average value p^* is calculated as follows:

$$p^* = \frac{(1*14)+(3*18.33)+(2*30.83)+(1*32.66)}{1+3+2+1} = 23.33$$

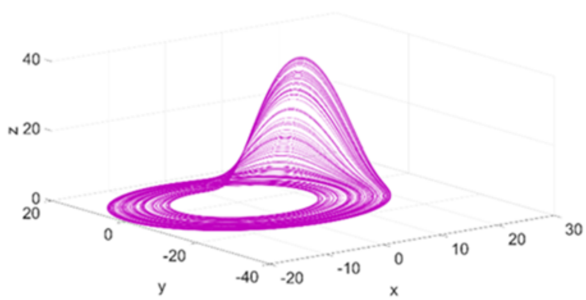
The underlying attractors Ap_1, Ap_2, Ap_3 and Ap_4 are chaotic (see Figures 2 (a) - (d)), while the "switched" solution is a stable cycle (see Figure 2 (e)). The "switched" solution (red color) approximates the "averaged" solution (blue color) corresponding to the p^* (see Figure 2(g)). Paradoxically, the PS scheme for N = 4 leads to the following chaos control-like:

$$chaos_1 + chaos_2 + chaos_3 + chaos_4 = order \qquad [9]$$

where $chaos_1$, $chaos_2$, $chaos_3$ and $chaos_4$ are the chaotic behaviors Ap_1, Ap_2, Ap_3 and Ap_4, while *order* is the stable cycle A^S from the PS algorithm.

Fig. 2. Chaos control in fractional order Rössler system with p^*=23.33, PS scheme $[m_1p_1, m_2p_2, m_3p_3, m_4p_4]$ and $P_4 = \{14, 18.33, 30.83, 32.66\}$ (a) – (d) Chaotic attractors Ap_1, Ap_2, Ap_3 and Ap_4 (e) "Switched" solution A^S (f) "Averaged" solution A^* (g) Over-plots of A^S (red) and A^* (blue) (h) – (j) Over-plot of time series of x, y and z of A^S and A^*.

(a)



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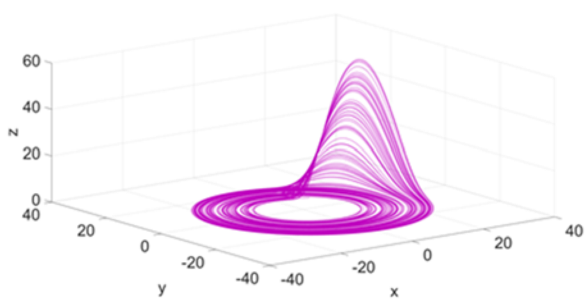

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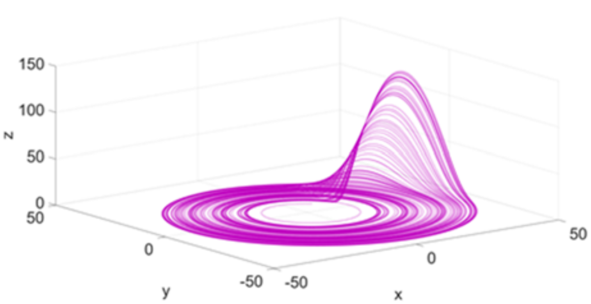

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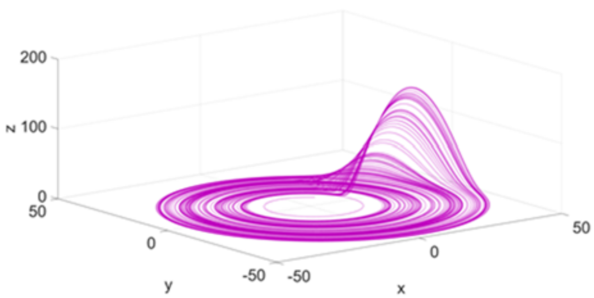

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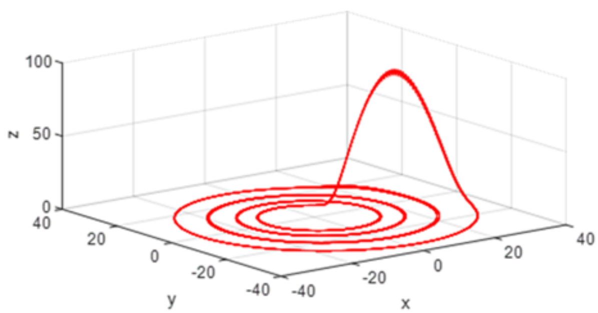

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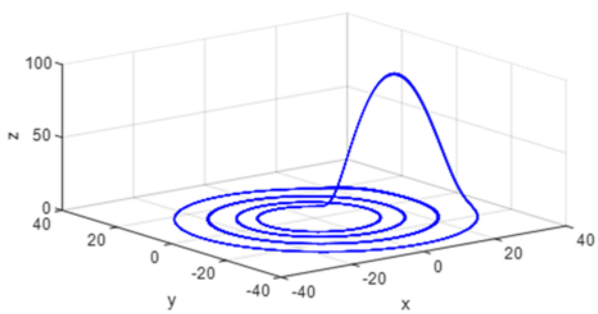

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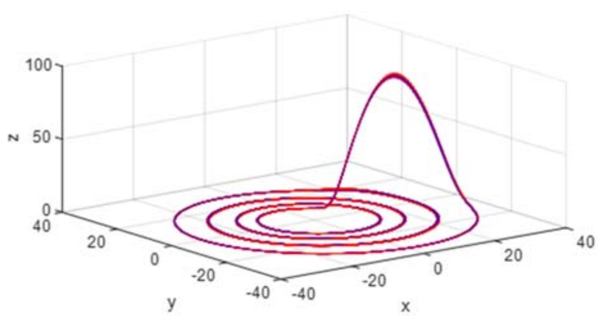

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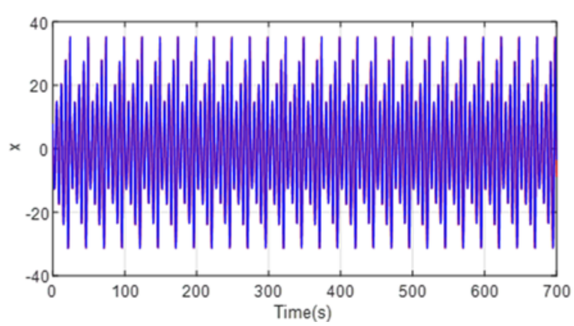

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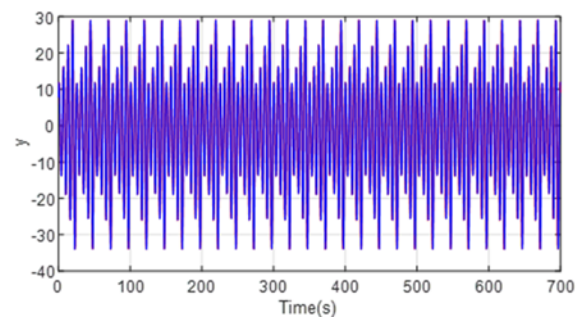

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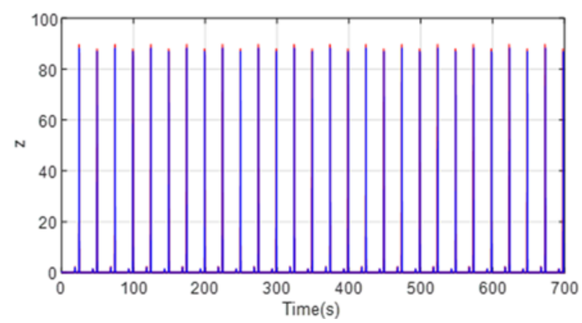


Table 1 shows the values of the LEs of the chaotic attractors Ap_1, Ap_2, Ap_3 and Ap_4 and the stable cycle A^* corresponding to the average parameter p^*, approximated by the "switched" solution of the PS scheme. It is noted in A^* that all the three LEs are less than zero, which is an indication of a stabilized system.

Table 1. Lyapunov exponents of unstable cycles $A_4 = \{A_{p1}, A_{p2}, A_{p3}, A_{p4}\}$ and stable cycle A^* for fractional order Rössler system

Lyapunov exponents	$\boldsymbol{Ap_1}$	$\boldsymbol{Ap_2}$	$\boldsymbol{Ap_3}$	$\boldsymbol{Ap_4}$	$\boldsymbol{A^*}$
LE1	0.0599	0.0795	0.0784	0.0713	-0.0034
LE2	0.0011	0.0010	0.0011	0.0025	-0.0119
LE3	-1.8024	-1.7983	-1.6341	-1.6940	-2.6071

3.2 Case 2: Chaos Anti-control

In this case of anti-control, the focus is on the chaotic regions of the bifurcation diagram in Figure 1. The system is chaotic at several values on control parameter p, but $p=33.32$ is selected for this investigation. To synthesize unstable attractor based on PS scheme $[m_1p_1, m_2p_2, m_3p_3, m_4p_4]$, let $p_1 = 23.28$, $p_2 = 23.33$, $p_3 = 23.40$, and $p_4 = 49.98$ represent four stable orbits of the fractional order Rössler system, while $m_1 = 2$, $m_2 = 2$, $m_3 = 1$, and $m_4 = 3$ are the associated weights. The average value $p*$ is calculated as follows:

$$p^* = \frac{(2 * 23.28) + (2 * 23.33) + (1 * 23.40) + (3 * 49.98)}{1 + 3 + 2 + 1} = 33.32$$

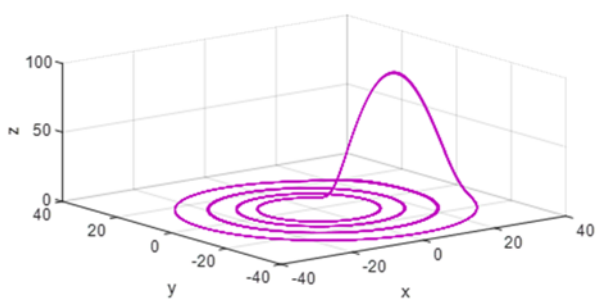
The underlying attractors Ap_1, Ap_2, Ap_3 and Ap_4 are stable (see Figures 3 (a) - (d)), while the "switched" solution is an unstable attractor (see Figure 3(e)). However, it is well-observed in the time series that the attractors do not match completely (see Figures 3 (h) - (j)) as good as those in chaos control case. This is because chaotic attractors can only be approximated theoretically after a sufficiently long period of time [11].
Paradoxically, the PS scheme for $N = 4$ leads to the following chaos anti-control manner:

$$order_1 + order_2 + order_3 + order_4 = chaos \qquad (10)$$

where $order_1$, $order_2$, $order_3$ and $order_4$ are the chaotic behaviors Ap_1, Ap_2, Ap_3 and Ap_4, while *chaos* is the unstable cycle A^S from the PS algorithm.

Fig. 3. Chaos anti-control in fractional order Rössler system with p^*=33.32, PS scheme $[m_1p_1, m_2p_2, m_3p_3, m_4p_4]$ and $P_4 = \{23.28, 23.33, 23.40, 49.98\}$ (a) – (d) Chaotic attractors Ap_1, Ap_2, Ap_3 and Ap_4 (e) "Switched" solution A^S (f) "Averaged" solution A^* (g) Over-plot of A^S (red) and A^* (blue) (h) – (j) Over-plots of time series of x, y and z of A^S and A^*.

(a)



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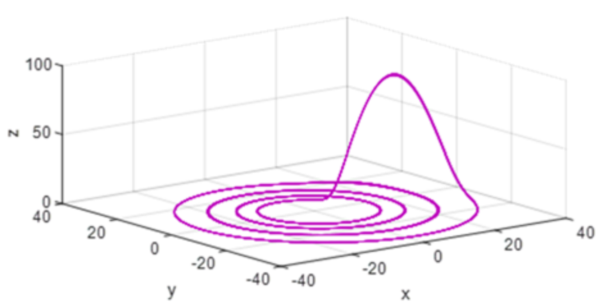

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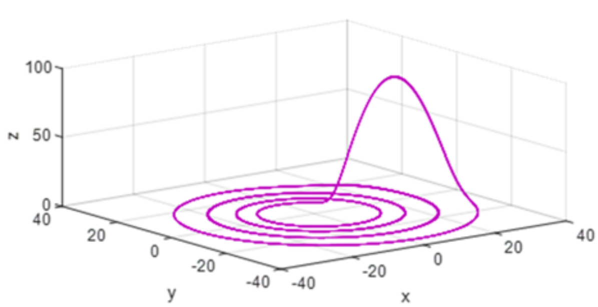

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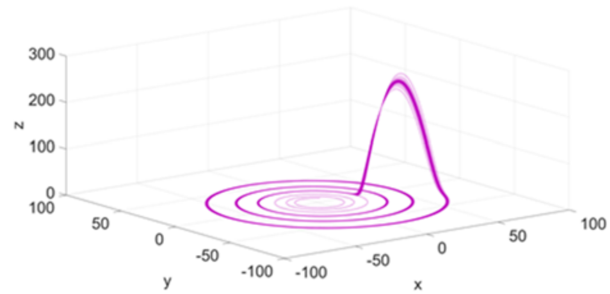

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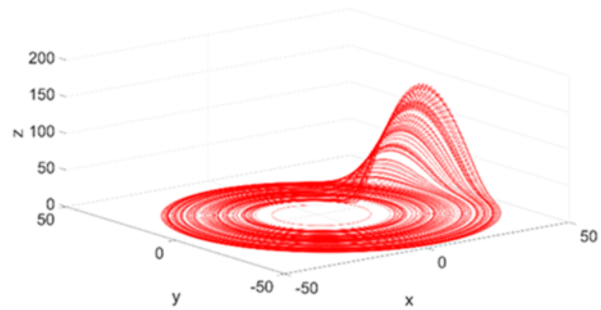

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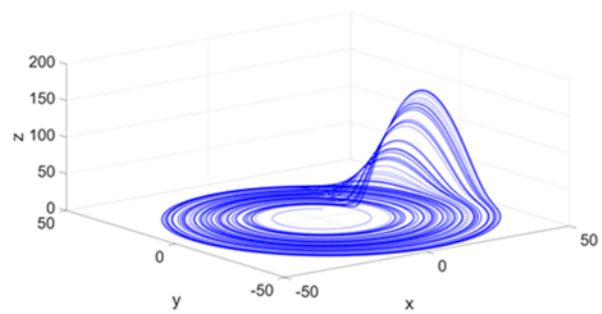

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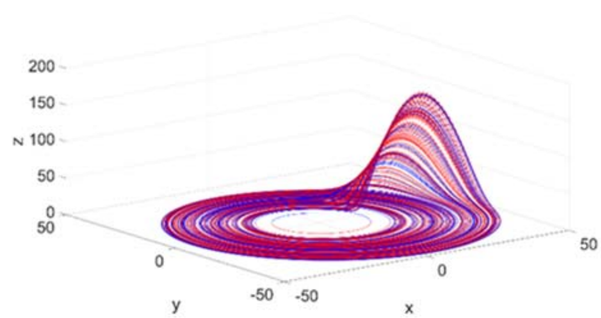

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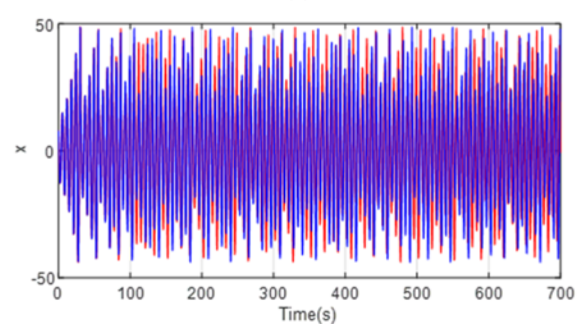

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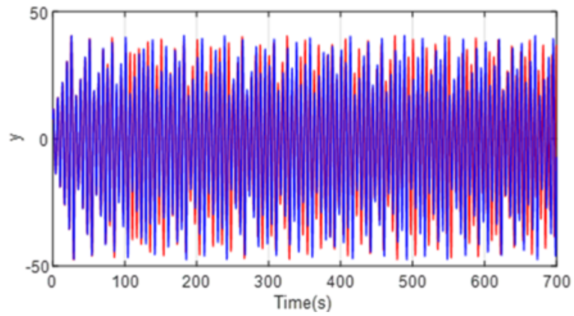

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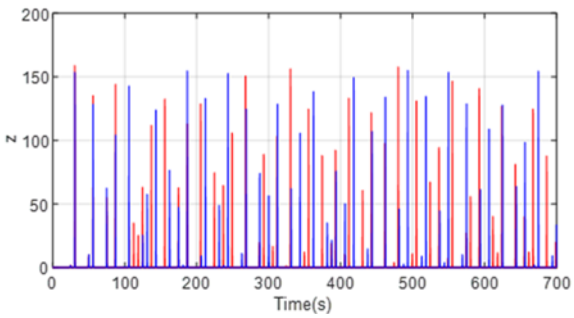

In Table 2 is found the values of the LEs of the stable attractors Ap_1, Ap_2, Ap_3 and Ap_4 and the synthesized unstable attractor A^* corresponding to the average parameter p^*. The positive maximum LE of A^* shows that the synthesized attractor is chaotic.

Table 2. Lyapunov exponents of stable cycles $A_4 = \{A_{p1}, A_{p2}, A_{p3}, A_{p4}\}$ and unstable cycle A^* for fractional order Rössler system

Lyapunov exponents	Ap_1	Ap_2	Ap_3	Ap_4	A^*
LE1	-0.0048	-0.0034	-0.0004	-0.0002	0.0595
LE2	-0.0116	-0.0119	-0.0133	-0.0048	0.0030
LE3	-2.6121	-2.6071	-2.6079	-2.2763	-2.3051

4. CONCLUSION

In this work, parameter switching technique was presented as a suitable technique to suppress chaos as well as anti-control of chaos in fractional order Rössler system, based on switching a set of chosen parameter values in a deterministic manner. The results are verified using phase portraits, time series and Lyapunov exponents. This contribution shows a great potential for PS application as a component in designing secure chaotic communication system, whereby chaos is used for encryption while the transmitted information is recovered by stabilizing the system by parameter switching technique.

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